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A proof that convection in a porous vertical slab is stable

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One of the motivations for study of convection in a porous medium comes from geophysics (see, for example, Elder 1966). Another stems from the fact that convection in a porous medium is often very similar to convection in a viscous fluid, yet the governing equations are of a lower order and so more amenable to study. The viscous-fluid analogue of the problem studied in this paper is of interest in connexion with building insulation involving an unventilated air gap. In that case (see comments of Gill & Davey 1969, §7) the equilibrium situation is unstable if, for a given temperature difference across the gap, the width of the gap is above a certain value. Then a cellular motion occurs which will increase the heat transfer considerably. It is shown below, however, that if the gap is filled with a porous medium (fibre glass is often used in the building industry), the corresponding equilibrium situation is always stable.

Consider the natural convection of a fluid of kinematic viscosity ν, density ρ and specific heat c which saturates a vertical slab of a porous medium of permeability k and thickness L. The slab is assumed to have infinite height. Suppose the convection is induced by holding the vertical boundaries of the slab at different temperatures, the difference being ΔT. Then the motion is governed by a single parameter, the Rayleigh number A given by

$$A = \rho c k \gamma g \Delta T L / K_m \nu,$$

where γ is the coefficient of cubical expansion, g the acceleration due to gravity and K_m the thermal conductivity of the fluid-saturated medium. Since k is proportional to the square of the length scale of the voids in the medium, typical values of A are very much smaller than the corresponding values for a viscous fluid alone. The equations (see Wooding 1957, 1960 and Elder 1966), in non-dimensional form, are

$$\left.\begin{aligned} u &= -p_x, \\ w &= -p_z + T, \\ u = -\psi_z, &\quad w = \psi_x, \\ A(T_t + uT_x + wT_z) &= \nabla^2 T, \end{aligned}\right\} \tag{1}$$

where the Boussinesq approximation has been used and units of length, velocity, temperature, pressure and time are chosen respectively as

$$L, \quad k\gamma g \Delta T/\nu, \quad \Delta T, \quad \rho\gamma g \Delta T L \quad \text{and} \quad E\nu L/k\gamma g \Delta T.$$

ρcE (see Wooding 1960) is the heat capacity per unit volume of the fluid-saturated medium and the zero of the temperature scale has been set at the mean of the two boundary temperatures. The components (u, w) of the volume flow vector correspond to rectangular co-ordinates (x, z) chosen so that the z-axis points vertically upwards and such that the boundaries at temperature $T = \pm\frac{1}{2}$ are given respectively by $x = \pm\frac{1}{2}$. The steady solution of the problem satisfying the requirement that the net vertical volume flux be zero is

$$w = W(x) = x, \quad T = \Theta(x) = x. \tag{2}$$

Consider now the stability of this equilibrium to small disturbances, the perturbations to the stream function ψ and temperature T being respectively

$$\phi(x)\exp[i\alpha(z-ct)], \quad \theta(x)\exp[i\alpha(z-ct)],$$

where α is a real constant and $c = c_r + ic_i$ a complex constant. Since Squire's theorem is obviously valid, only two-dimensional disturbances need be considered. Using (1), the following stability equations are derived

$$\phi'' - \alpha^2\phi = \theta', \tag{3}$$

$$\theta'' - \alpha^2\theta = i\alpha A[(W-c)\theta - \Theta'\phi], \tag{4}$$

the boundary conditions being that

$$\phi = \theta = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}.$$

It will now be shown that the equilibrium given by (2) is stable. In this case, equation (4) gives

$$\phi = (x-c)\theta - (\theta'' - \alpha^2\theta)/i\alpha A$$

and substitution in (3) leads to the equation

$$\theta^{iv} - 2\alpha^2\theta'' + \alpha^4\theta - i\alpha A\{[(x-c)\theta']' - \alpha^2(x-c)\theta\} = 0.$$

Multiplying this by the complex conjugate of θ, integrating from $x = -\frac{1}{2}$ to $x = \frac{1}{2}$ and taking the real part, leads to the relation

$$\alpha A c_i \int_{-\frac{1}{2}}^{\frac{1}{2}} \{|\theta'|^2 + |\alpha\theta|^2\}\,dx = -\int_{-\frac{1}{2}}^{\frac{1}{2}} |\theta'' - \alpha^2\theta|^2\,dx$$

which shows that $c_i < 0$, that is, that the equilibrium is stable.

Why, then, is the porous-medium equilibrium stable when the analogous viscous-fluid equilibrium is unstable? There is a result about convection in a viscous fluid which suggests an answer. Gill & Davey (1969), in studying instabilities of a particular equilibrium, found (§5) that, although the full disturbance equations had unstable solutions, the equation obtained by omitting the inertial terms had only stable solutions. Thus the instability depended in some way on the existence of inertial effects. Now, in a porous medium, velocities are so small that inertial effects are normally ignored, as they are in (1). Gill & Davey's result suggests that this absence of inertial effects is responsible for the absence of instability in the porous-medium case. In other words, it is suggested that the principal stabilizing effect of placing fibre glass (say) in a cavity is due to the great reduction in the significance of inertia effects.

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